ON PSEUDO-RIEMANNIAN MANIFOLDS WITH RECURRENT CONCIRCULAR CURVATURE TENSOR

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Abstract. It is proved that every concircularly recurrent manifold must be necessarily a recurrent manifold with the same recurrence form.

1. Introduction

For a pseudo-Riemannian manifold (M,g), by T,U,V,W,X,Y,Z will be denoted arbitrary smooth vector fields on M, and for the Riemann curvature operator $\mathcal{R}$ and the Riemann curvature $(0,4)$-tensor R, we assume the following conventions

$$\mathcal{R}(X,Y) = [\nabla_X,\nabla_Y] - \nabla_{[X,Y]} = \nabla^2_{X,Y} - \nabla^2_{Y,X},$$

$$R(W,X,Y,Z) = g(\mathcal{R}(W,X)Y,Z),$$

where ∇ indicates the covariant derivative with respect to the Levi-Civita connection, and ∇^2 is the second covariant derivative,

$$\nabla^2_{X,Y} = \nabla_X\nabla_Y - \nabla_{\nabla_X Y}.$$

Pseudo-Riemannian manifolds are assumed to be connected.

A pseudo-Riemannian manifold (M,g) is said to be recurrent [13, 16] if its Riemann curvature operator $\mathcal{R}$ is recurrent, that is, $\mathcal{R}$ is non-zero and its covariant derivative $\nabla\mathcal{R}$ satisfies the condition

$$\nabla\mathcal{R} = \lambda\otimes\mathcal{R} \tag{1}$$

for a certain 1-form λ (the recurrence form).

For a pseudo-Riemannian manifold (M,g), the concircular curvature tensor field $\mathcal{C}$ is defined as

$$\mathcal{C} = \mathcal{R} - \frac{r}{n(n-1)}\mathcal{G}, \tag{2}$$

where $n = \dim M$, r is the scalar curvature and $\mathcal{G}$ is the curvature like operator defined as

$$\mathcal{G}(X,Y)Z = g(Y,Z)X - g(X,Z)Y.$$

The tensor $\mathcal{C}$ is an invariant of the concircular transformations which have many important geometric and algebraic applications; see [19, 15, 8, 7], etc. For our purpose, we recall only two facts: (1) when $\dim M = 2$, then $\mathcal{C} = 0$ and such a manifold realizes the condition $\nabla\mathcal{R} = \lambda\otimes\mathcal{R}$ with $\lambda = \nabla(\ln|r|)$ at each point at which $\mathcal{R}\neq 0$; (2) when $\dim M \geqslant 3$, $\mathcal{C} = 0$ if and only if the manifold is of constant sectional curvature.

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A pseudo-Riemannian manifold (M, g) is said to be concircularly recurrent if its concircular curvature tensor $\mathcal{C}$ is recurrent, that is, $\mathcal{C}$ is non-zero and its covariant derivative $\nabla\mathcal{C}$ satisfies the condition

$$\nabla\mathcal{C} = \lambda \otimes \mathcal{C} \tag{3}$$

for a certain 1-form λ. For a concircularly recurrent manifold, $n = \dim M \geqslant 3$ and $\mathcal{C} \neq 0$ at every point of M.

It is obvious that a recurrent manifold is concircularly flat ($\mathcal{C} = 0$) or concircularly recurrent with the same recurrence form. The purpose of the presented paper is to prove the following theorem:

Theorem. *Every concircularly recurrent manifold is necessarily a recurrent manifold with the same recurrence form.*

The above theorem seems to be very important since concircularly recurrent manifolds were studied by many authors, see [2, 3, 4, 5, 6, 10, 11, 14], etc. In view of our theorem, many results and proofs occured in some of the listed papers can be simplified, sometimes radically.

Hint. *With the recurrence notion used in the present paper, we follow Y.-C. Wong* [17, 18]*; cf. also* [9]. *Due to Theorem 3.8 from* [17]*, on a connected differentiable manifold endowed with an affine connection, any recurrent tensor field has no zeros.*

2. Proof of the theorem

For a concirculary recurrent manifold, as a consequence of (2) and (3), we claim that the Riemann curvature operator satisfies the following condition

$$\nabla\mathcal{R} = \lambda \otimes \mathcal{R} + \mu \otimes \mathcal{G}, \tag{4}$$

where the 1-form λ is the same as in (3) and the 1-form μ is given by

$$\mu = \frac{1}{n(n-1)}(dr - r\lambda). \tag{5}$$

Conversely, if a pseudo-Riemannian manifold satisfies (4) with certain 1-forms λ and μ, then μ must be of the form (5), and (3) must be realized. The equivalence of (3) and (4) has already been noticed in [2]. It is worth to remark that if $\mu = 0$, then obviously (1) holds so that the concircular recurrence reduces to the recurrence.

Before we start with the proof, it will be useful to recall the famous Walker identity (see [16, Lemma 1]) stating that for any pseudo-Riemannian manifold it holds

$$\begin{aligned}(\nabla^2_{U,V}R - \nabla^2_{V,U}R)(W,X,Y,Z) + (\nabla^2_{W,X}R - \nabla^2_{X,W}R)(Y,Z,U,V)\\ + (\nabla^2_{Y,Z}R - \nabla^2_{Z,Y}R)(U,V,W,X) = 0.\end{aligned}$$

Rewrite the Walker identity in the following form, which will be more convenient for us

$$(\mathcal{R}(U,V)R)(W,X,Y,Z) + (\mathcal{R}(W,X)R)(Y,Z,U,V) + (\mathcal{R}(Y,Z)R)(U,V,W,X) = 0. \tag{6}$$

We are going to reach the assertion of the Theorem in the following three steps.

However, we would like to add that the closedness of the form λ (see the first step below) has already been proved in [2]. We have included it only for the completness of the whole of the proof.

Step 1. *For a concircularly recurrent manifold, the recurrence form λ is closed.*

Proof. Let C denotes the $(0,4)$-tensor related to $\mathcal{C}$ by

$$C(W,X,Y,Z) = g(\mathcal{C}(W,X)Y,Z).$$

Using (3), we have $\nabla_V C = \lambda(V)C$, and next

$$\nabla^2_{U,V}C = ((\nabla_U\lambda)(V) + \lambda(U)\lambda(V))C.$$

Therefore,

$$\mathcal{R}(U,V)C = \nabla^2_{U,V}C - \nabla^2_{V,U}C = 2d\lambda(U,V)C,$$

where we have used the formula

$$d\lambda(U,V) = \frac{1}{2}((\nabla_U\lambda)(V) - (\nabla_V\lambda)(U)).$$

Consequently, we obtain

$$\begin{aligned}(7)\qquad & (\mathcal{R}(U,V)C)(W,X,Y,Z) + (\mathcal{R}(W,X)C)(Y,Z,U,V)\\ &+ (\mathcal{R}(Y,Z)C)(U,V,W,X) = 2d\lambda(U,V)C(W,X,Y,Z)\\ &+ 2d\lambda(W,X)C(Y,Z,U,V) + 2d\lambda(Y,Z)(Y))C(U,V,W,X),\end{aligned}$$

On the other hand, note that from (2) it follows that

$$C = R - \frac{r}{n(n-1)}G, \tag{8}$$

where G is the curvature like $(0,4)$-tensor related to $\mathcal{G}$ by

$$G(W,X,Y,Z) = g(\mathcal{G}(W,X)Y,Z).$$

Using (8) and the identities $\mathcal{R}(U,V)r = 0$, $\mathcal{R}(U,V)G = 0$, we find

$$\begin{aligned}(9)\qquad & (\mathcal{R}(U,V)C)(W,X,Y,Z) + (\mathcal{R}(W,X)C)(Y,Z,U,V)\\ &+ (\mathcal{R}(Y,Z)C)(U,V,W,X) = (\mathcal{R}(U,V)R)(W,X,Y,Z)\\ &+ (\mathcal{R}(Y,Z)R)(U,V,W,X) + (\mathcal{R}(W,X)R)(Y,Z,U,V).\end{aligned}$$

Therefore, from (6), (9) and (7), we have

$$d\lambda(U,V)C(W,X,Y,Z) + d\lambda(W,X)C(Y,Z,U,V) + d\lambda(Y,Z)C(U,V,W,X) = 0.$$

Since $C \neq 0$, by applying the famous Walker lemma ([16, Lemma 2]), we obtain from the last formula

$$d\lambda = 0, \tag{10}$$

completing the proof of Step 1. □

Step 2. *A concircularly recurrent manifold is semisymmetric.*

Proof. Using (4), for the first covariant derivative of the Riemann curvature $(0,4)$-tensor R, we find

$$\nabla_V R = \lambda(V)R + \mu(V)G, \tag{11}$$

and for the second covariant derivative,

$$\nabla^2_{U,V}R = ((\nabla_U\lambda)(V) + \lambda(U)\lambda(V))R + ((\nabla_U\mu)(V) + \mu(U)\lambda(V))G.$$

Hence, using also (10), we obtain

$$\mathcal{R}(U,V)R = \nabla^2_{U,V}R - \nabla^2_{V,U}R = 2(d\mu + \mu\wedge\lambda)(U,V)G. \tag{12}$$

Applying the above identity into the Walker identity (6), we obtain

$$\begin{aligned}(d\mu + \mu\wedge\lambda)(U,V)G(W,X,Y,Z) + (d\mu + \mu\wedge\lambda)(W,X)G(Y,Z,U,V)\\ + (d\mu + \mu\wedge\lambda)(Y,Z)G(U,V,W,X) = 0.\end{aligned}$$

Since $G \neq 0$, by applying the famous Walker lemma, we obtain

$$d\mu + \mu\wedge\lambda = 0.$$

The last relation reduces (12) to

$$\mathcal{R}(U,V)R = 0, \tag{13}$$

which is just the semisymmetry (cf. [1]). □

Step 3. *A concircularly recurrent manifold is recurrent.*

Proof. Since $\mathcal{R}(U,V)$ is a derivation of the tensor algebra on M (cf. e.g. [12]), we have

$$\begin{aligned}(\mathcal{R}(U,V)R)(W,X,Y,Z) = &- R(\mathcal{R}(U,V)W,X,Y,Z) - R(W,\mathcal{R}(U,V)X,Y,Z)\\ &- R(W,X,\mathcal{R}(U,V)Y,Z) - R(W,X,Y,\mathcal{R}(U,V)Z).\end{aligned}$$

Hence, having the semisymmetry condition (13), we obtain

$$\begin{aligned}&R(\mathcal{R}(U,V)W,X,Y,Z) + R(W,\mathcal{R}(U,V)X,Y,Z)\\ &\quad + R(W,X,\mathcal{R}(U,V)Y,Z) + R(W,X,Y,\mathcal{R}(U,V)Z) = 0.\end{aligned} \tag{14}$$

Now, differentiating the above equality covariantly, we get

$$\begin{aligned}&(\nabla_T R)(\mathcal{R}(U,V)W,X,Y,Z) + R((\nabla_T\mathcal{R})(U,V)W,X,Y,Z)\\ &+ (\nabla_T R)(W,\mathcal{R}(U,V)X,Y,Z) + R(W,(\nabla_T\mathcal{R})(U,V)X,Y,Z)\\ &+ (\nabla_T R)(W,X,\mathcal{R}(U,V)Y,Z) + R(W,X,(\nabla_T\mathcal{R})(U,V)Y,Z)\\ &+ (\nabla_T R)(W,X,Y,\mathcal{R}(U,V)Z) + R(W,X,Y,(\nabla_T\mathcal{R})(U,V)Z)\\ &= 0.\end{aligned}$$

Hence, by applying (4) and (11), we find

(15)

$$\begin{aligned}&(\lambda(T)R + \mu(T)G)(\mathcal{R}(U,V)W,X,Y,Z) + R((\lambda(T)\mathcal{R} + \mu(T)\mathcal{G})(U,V)W,X,Y,Z)\\ &+ (\lambda(T)R + \mu(T)G)(W,\mathcal{R}(U,V)X,Y,Z) + R(W,(\lambda(T)\mathcal{R} + \mu(T)\mathcal{G})(U,V)X,Y,Z)\\ &+ (\lambda(T)R + \mu(T)G)(W,X,\mathcal{R}(U,V)Y,Z) + R(W,X,(\lambda(T)\mathcal{R} + \mu(T)\mathcal{G})(U,V)Y,Z)\\ &+ (\lambda(T)R + \mu(T)G)(W,X,Y,\mathcal{R}(U,V)Z) + R(W,X,Y,(\lambda(T)\mathcal{R} + \mu(T)\mathcal{G})(U,V)Z)\\ &= 0.\end{aligned}$$

Let us assume that $\mu \neq 0$ at a certain point of M. At this point, using (14), the equality (15) can be reduced to

$$\begin{aligned}&G(\mathcal{R}(U,V)W,X,Y,Z) + R(\mathcal{G}(U,V)W,X,Y,Z)\\ &+ G(W,\mathcal{R}(U,V)X,Y,Z) + R(W,\mathcal{G}(U,V)X,Y,Z)\\ &+ G(W,X,\mathcal{R}(U,V)Y,Z) + R(W,X,\mathcal{G}(U,V)Y,Z)\\ &+ G(W,X,Y,\mathcal{R}(U,V)Z) + R(W,X,Y,\mathcal{G}(U,V)Z) = 0.\end{aligned}$$

When using the definitions of the tensor G and the operator $\mathcal{G}$, the last equality takes the following form

$$\begin{aligned}
&g(V,W)R(U,X,Y,Z)-g(U,W)R(V,X,Y,Z)\\
&+g(V,X)R(W,U,Y,Z)-g(U,X)R(W,V,Y,Z)\\
&+g(V,Y)R(W,X,U,Z)-g(U,Y)R(W,X,V,Z)\\
&+g(V,Z)R(W,X,Y,U)-g(U,Z)R(W,X,Y,V)=0.
\end{aligned}$$

Contracting the above with respect to the pair of arguments V,W (this means that we take the trace $\mathrm{Trace}_g\{(V,W)\mapsto ...\}$, where the dots stand for the relation to be traced), we obtain

$$\begin{aligned}
&(n-2)R(U,X,Y,Z)+R(Y,X,U,Z)+R(Z,X,Y,U)\\
&\quad+g(U,Y)S(X,Z)-g(U,Z)S(X,Y)=0,
\end{aligned}$$

which with the help of the first Bianchi identity becomes

$$(n-1)R(U,X,Y,Z)+g(U,Y)S(X,Z)-g(U,Z)S(X,Y)=0, \tag{16}$$

S being the Ricci curvature tensor. Contracting the obtained relation with respect to the pair of arguments X,Z, we get the Einstein condition $S=(r/n)g$, which applied to (16), gives us $R=(r/(n(n-1)))G$, or equivalently $C=0$, contradicting our assumption. Therefore, $\mu=0$ at every point of M, and consequently, the concircular recurrence reduces to the recurrence. □

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